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Stability of Motion on Three-Dimensional Halo Orbits

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This paper presents a study regarding the stability of the three-dimensional halo orbits. The origin of the orbits is in the vicinities of the libration points L_1, L_2, L_3, as they are defined in the three-body problem. The theory develops points on the possibility of determining the stable and unstable domains of the motion for values of the parameter $\mu = m_2/(m_1 + m_2)$ between 0 and 1 (where m_1 and m_2 are the primary masses). A group of theorems regarding differential equations with periodic coefficients is used to determine in the end the stability conditions for the motion.The paper also presents corrections to the initial conditions so that the orbits under study belong to the periodic class. A numerical application confirms an earlier result of Goudas.

I. Introduction

THE restricted three-body problem is a model often used in space dynamic studies.[1–6] A major interest includes the motion of an infinitesimal particle under the gravitational influence of two finite bodies.[7]

The present study regards the motion (resulting from some particular initial conditions) on three-dimensional periodic halo orbits. The term halo was introduced by Farquhar and Kamel[8] in 1973. Farquhar[8] used the Lindstedt–Poincaré method to determine the analytical solutions for the quasiperiodic orbits of the L_2 point in the Earth-moon system.

The calculations have shown the instability of this orbit family.

The ISEE satellite was launched on an unstable halo orbit around the L_1 libration point of the sun-Earth system. In 1979 Breakwell and Brown[10] extended the numerical calculations of Farquhar and Kamel[8] for the L_2 family of periodic orbits and obtained a new class of linear orbits placed in the vicinity of the moon. They also calculated the family for L_1 when it is moving toward the moon. A stable domain placed approximately at the middle of the distance between the moon and the L_1 and L_2 libration points was determined for those points as well.

New cases for the variation of mass ratio μ between 0 and 1 were considered in Ref. 11. The numerical integration showed that, for a given mass ratio, the halo orbits of the collinear libration, L_1 and L_2, have a comparable magnitude and increase together with μ. The periods of the majority of the orbits are decreasing when approaching the nearest mass. In the vicinity of L_1 the stable orbits are nearer to the libration point when μ is increasing and they disappear when $\mu = 0.0573$. For a sufficiently large mass ratio, in the vicinity of L_3, there is an intersection frontier of the stability domains.

The stability of motion on the orbits of the equilateral libration points L_4 and L_5, using the theory of the stability in critical cases, is presented in Ref. 12.

The present paper avoids the difficulties created by the numerical integration in the previous studies by providing an analytical method[15,16] for the analysis of the motion on the periodic three-dimensional halo orbits.

This is fulfilled by a stability analysis of the solutions of the nonlinear equations of the motions around the collinear libration points.

II. Equations of Motion

The determination of the motion of an infinitesimal body (a body having negligible mass by comparison with the attraction center or centers) in the field of two finite mass bodies m_1 and m_2 is known as the restricted problem of three bodies.

The finite masses, presumed to be material points, are moving around their common center of mass, each being under the gravitational influence of the other. The reference frame is a rotating one with the x axis on the m_1-m_2 line and the origin at the barycenter. The masses m_1 and m_2 move in the xy plane and the z axis is orthogonal to this plane. The infinitesimal body is moving in the three-dimensional system xyz. There are five equilibrium or libration points where the gravitational and centrifugal forces are equal. Three of these equilibrium points, L_1, L_2, and L_3, in line with the finite masses m_1 and m_2, are called collinear libration points. The other two, L_4 and L_5, are called equilateral libration points and form two equilateral triangles with the finite masses.

The ratio $\mu = m_1/(m_1 + m_2)$ is a fundamental parameter of the motion. As long as m_1 and m_2 are rotating around the barycenter, the five points maintain the same relative positions with respect to the finite masses for a given μ.

We adopt a dimensionless unit system as follows: the distance between m_1 and m_2 is 1, the sum of masses is 1, and the angular velocity is 1.

Under the previous assumptions the equations of motion can be written as

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y}, \qquad \ddot{z} = \frac{\partial U}{\partial z} \tag{1}$$

where

$$\begin{aligned} U &= \tfrac{1}{2}(x^2 + y^2) + (1-\mu)/r_1 + \mu/r_2 \\ r_1 &= [(x+\mu)^2 + y^2 + z^2]^{\frac{1}{2}} \\ r_2 &= [(x-1+\mu)^2 + y^2 + z^2]^{\frac{1}{2}} \end{aligned} \tag{2}$$

Using the notation

$$\begin{aligned} x &= x_1, & y &= x_2, & z &= x_3 \\ \dot{x} &= x_4, & \dot{y} &= x_5, & \dot{z} &= x_6 \end{aligned} \tag{3}$$

system (1) becomes

$$\begin{aligned} \dot{x}_1 &= x_4 = f_1 \\ \dot{x}_2 &= x_5 = f_2 \\ \dot{x}_3 &= x_6 = f_3 \\ \dot{x}_4 &= \frac{\partial U}{\partial x_1} + 2x_5 = f_4 \\ \dot{x}_5 &= \frac{\partial U}{\partial x_2} - 2x_4 = f_5 \\ \dot{x}_6 &= \frac{\partial U}{\partial x_3} = f_6 \end{aligned} \tag{4}$$

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The initial conditions for the differential equations of system (4) are given by

$$x_0 = x_1^0, \qquad y_0 = x_2^0, \qquad z_0 = x_3^0$$
$$\dot{x}_0 = x_4^0, \qquad \dot{y}_0 = x_5^0, \qquad \dot{z}_0 = x_6^0 \tag{5}$$

We presume that in the range

$$D: \left\{(t, x_i) \mid |t - \bar{t}_0| \le \alpha, \left|x_i - \bar{x}_i^0\right| \le \beta \qquad (i = 1, 2, \ldots, 6)\right\} \tag{6}$$

the function f_i admits continuous partial derivatives related to the $\{x_i\}$ variables set. If $x_i^0 \in D_0 \subset D$, then the solution

$$x_i = x_i\left(t; x_i^0\right) \tag{7}$$

admits continuous derivatives relative to the initial conditions,

$$z_i = \frac{\partial x_i}{\partial x_j^0} = \Phi_{ij} \tag{8}$$

and satisfies the linear and homogenous differential equation system

$$\dot{z}_i = \sum_{j=1}^{6} \frac{\partial f_i(t, x_1, \ldots, x_6)}{\partial x_j} z_j \qquad (i = 1, \ldots, 6) \tag{9}$$

The stability analysis of system (4) is equivalent to the stability study for the variation system solutions (9). If $\Phi(t, 0)$ represents a fundamental solution system for system

$$\dot{\Phi}(t, 0) = A(t)\Phi(t, 0) \tag{10}$$

where (9) then

$$\Phi(t, 0) = (\Phi_{ij}) = \left(\frac{\partial x_i}{\partial x_j^0}\right) \qquad (i, j = 1, \ldots, 6)$$

$$A(t) = \begin{bmatrix} 0 & E_3 \\ S(t) & 2K \end{bmatrix} \tag{11}$$

where 0 is the null matrix, E_3 is the third-order identity matrix, and

$$K = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad S(t) = \begin{bmatrix} \dfrac{\partial^2 U}{\partial x\, \partial x} & \dfrac{\partial^2 U}{\partial x\, \partial y} & \dfrac{\partial^2 U}{\partial x\, \partial z} \\ \dfrac{\partial^2 U}{\partial y\, \partial x} & \dfrac{\partial^2 U}{\partial y\, \partial y} & \dfrac{\partial^2 U}{\partial y\, \partial z} \\ \dfrac{\partial^2 U}{\partial z\, \partial x} & \dfrac{\partial^2 U}{\partial z\, \partial y} & \dfrac{\partial^2 U}{\partial z\, \partial z} \end{bmatrix} \tag{12}$$

System (10) includes 36 differential equations and the initial conditions are used for its integration:

$$\Phi(0, 0) = E \tag{13}$$

Note that matrix $S(t)$ is a symmetric matrix.

III. Three-Dimensional Halo Periodic Orbit

Let us consider C: $\{x = x(t); y = y(t); z = z(t)\}$. The parametric representation of the halo periodic orbit class obtained by the integration of the system of differential equations of motion (4) is invariant to the transformation:

$$t^* = -t, \qquad y^* = -y$$

Thus the solution of system (4) represented by the column vector

$$X(t) = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T \tag{14}$$

is symmetric relative to the plane (x, z) so that the initial conditions $X(0)$ are contained in this plane.

In this way we obtain

$$X(0) = [x_0, 0, z_0, 0, \dot{y}_0, 0]^T \tag{15}$$

The periodicity condition is given by

$$X(0) = X(\omega) \tag{16}$$

where $\omega = T^*$ is the period of motion. Considering the symmetry of this orbit related to the plane (x, z), the motion analysis will be sufficient for $t \in [0, T^*/2]$, where $t_1 = T^*/2$ represents the time when the orbit again intersects the plane (x, y) after $t_0 = 0$.

Note that

$$\operatorname{sign} \dot{y}(0) = -\operatorname{sign} \dot{y}(T^*/2) \tag{17}$$

In order to determine an admissible variation range of the initial conditions so that orbits are periodic, it will be necessary to state a set of initial conditions, the correction of the others being obtained in relation to the final initial conditions.

Let

$$X = X(X(0), t) \tag{18}$$

be the solution of the differential system of equations (4). The variation of the solution obtained while modifying the initial conditions during the time $t = T^*/2$ can be written as

$$(\delta X)^T = [\Phi](\delta X_0]^T + \left(\frac{\partial X}{\partial t}\right)^T \left(\delta \frac{T^*}{2}\right) \tag{19}$$

where we denote

$$(\delta X)^T = [\delta x, \delta y, \delta z, \delta \dot{x}, \delta \dot{y}, \delta \dot{z}]^T$$
$$(\delta X_0)^T = [\delta x_0, 0, \delta z_0, 0, \delta \dot{y}_0, 0]^T$$
$$\left(\frac{\delta X}{\delta t}\right)^T = [\dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}]^T \tag{20}$$
$$[\Phi] = (\Phi_{ij}) \qquad (i, j = 1, 2, \ldots, 6)$$

Presuming x_0 fixed, the other corrections of the initial data δz_0 and $\delta \dot{y}_0$ will also be determined depending on the initial conditions $(\delta X)^T$. It has to be taken into account that the final state belongs to the xz plane having the implication that

$$\delta(T^*/2) = -(\Phi_{23}\, \delta z_0 + \Phi_{25}\, \delta \dot{y}_0)/\dot{y} \tag{21}$$

Changing the value given by (21), expression (19) will become

$$(\delta X)^T = \left[(\Phi_{i3}, \Phi_{i5}) - \frac{1}{\dot{y}}\left(\frac{\partial X}{\partial t}\right)^T (\Phi_{23}, \Phi_{25})\right]\begin{bmatrix} \delta z_0 \\ \delta \dot{y}_0 \end{bmatrix} \qquad (i = 1, \ldots, 6) \tag{22}$$

Explicitly evaluating Eq. (22), it follows that

$$\begin{bmatrix} \delta \dot{x}_1 \\ \delta \dot{z}_1 \end{bmatrix} = \left(\begin{bmatrix} \Phi_{43} & \Phi_{45} \\ \Phi_{63} & \Phi_{65} \end{bmatrix} - \frac{1}{\dot{y}_1}\begin{bmatrix} \ddot{x}_1 \\ \ddot{z}_1 \end{bmatrix}[\Phi_{13} \quad \Phi_{25}]\right)\begin{bmatrix} \delta z_0 \\ \delta \dot{y}_0 \end{bmatrix} \tag{23}$$

where the index 1 means final values.

Observe that

$$\delta \dot{x}_1 = -\dot{x}_1 \qquad \delta \dot{z}_1 = -\dot{z}_1 \tag{24}$$

Integrating the equation system for the interval $t_1 = T^*/2$ yields $\ddot{x}_1, \ddot{x}_1, \dot{y}_1, \dot{z}_1$, which allows the determination of the initial data from Eq. (23). Integration should be continued as long as the results do not reach the lowest values for which y changes its sign, as calculated from relation (17).

Numerical integration reveals the ranges $|y_1| < 10^{-11}$ and $|\dot{x}_1, \dot{z}_1| < 10^{-3}$ for which orbits are periodic.[11] All developments were performed under the circumstances of keeping x fixed. The similar calculation that will be done for z_0 and $\dot{y}_0$ fixed can be considered as an assumption that will lead to the annulling of δz_0 and $\delta \dot{y}_0$, respectively.

IV. Monodromy Matrix of Variation System

Writing Eq. (9) in matrix form yields

$$\frac{dZ}{dt} = A(t)Z \tag{25}$$

Taking into account expression (11) for the matrix $A(t)$ and the periodicity property of the vector $X(t)$ shown in relation (16), it follows that

$$A(t+\omega) = A(t) \tag{26}$$

The solution $Z(t; t_0, Z_0)$ will define, for the t and t_0 fixed, a linear transformation

$$\tau\colon \Phi(t; t_0)Z_0 = Z(t; t_0, Z_0) \tag{27}$$

Relation (27) shows that each solution of system (25) belonging to the vector space is a linear combination of the fundamental solutions $\Phi(t, t_0)$ system.

The periodicity property of the system solutions can be written as

$$Z(t+\omega; s, Z_0) = Z(t; t_0, Z(t_0+\omega; s, Z_0)) \tag{28}$$

Relation (28) might be demonstrated choosing $s = t_0$. According to Eq. (27),

$$\begin{aligned} Z(t; t_0, Z(t_0+\omega; s, Z_0)) &= \Phi(t; t_0)Z(t_0+\omega; s, Z_0) \\ &= \Phi(t; t_0)\Phi(t_0+\omega; s)Z_0 \end{aligned} \tag{29}$$

By analogy

$$Z(t+\omega; s, Z_0) = \Phi(t+\omega; s)Z_0 \tag{30}$$

from which follows the equality

$$\Phi(t+\omega; s) = \Phi(t; t_0)\Phi(t_0+\omega; s) \tag{31}$$

For $s = t_0 + \omega$ equality (31) becomes

$$\Phi(t+\omega; t_0+\omega) = \Phi(t; t_0) \tag{32}$$

Considering $s = t_0 = 0$, it follows that

$$\Phi(t+\omega; 0) = \Phi(t; 0)\Phi(\omega; 0) \tag{33}$$

Denoting

$$\Phi(t; 0) = V(t) \tag{34}$$

relation (33) can be written as

$$V(t+\omega) = V(t)V(\omega) \tag{35}$$

The matrix $V(\omega)$ is the monodromy matrix of the variations system. Noting that this matrix is nonsingular, there is a matrix B such that

$$V(\omega) = e^{B\omega} \tag{36}$$

The behavior of the linear system with periodic coefficients depends on the eigenvalues of the matrix B. The characteristic equation of the monodromy matrix is given by

$$|V(\omega) - \rho E| = |e^{B\omega} - \rho E| = 0 \tag{37}$$

The characteristic roots λ_k of matrix B will be obtained using Eq. (37):

$$\lambda_k = (1/\omega)\ln \rho_k \tag{38}$$

where ρ_k are eigenvalues of the monodromy matrix.

We now show that for a class of differential systems with periodic coefficients the characteristic equation of the monodromy matrix is reciprocal.

The following theorem[14] will be employed for this purpose:

Theorem 1. Let us consider the class of periodic coefficients $\{\Gamma\}$:

$$\frac{dy}{dt} = P(t)y \tag{39}$$

with the property

$$HP(-t) + P(t)H = 0 \tag{40}$$

where

$$H = \begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix} \tag{41}$$

If a system $\gamma_i \in \Gamma$, then the characteristic equation of its monodromy matrix is reciprocal. Now we must demonstrate that the variation system belongs to the class Γ.

Due to the invariance of the motion equation system (4) to the transformation σ: $(t, y) \to (-t, -y)$, it follows that the solution of system (4) is a function of the variable $-t$, given by $x(-t), -y(-t), z(-t)$. Thus it follows that

$$X(-t) = \left[x(-t), y(-t), z(-t), \frac{dx(-t)}{d(-t)}, \frac{dy(-t)}{d(-t)}, \frac{dz(-t)}{d(-t)}\right]^T \tag{42}$$

Taking into account that $x(-t), -y(-t), z(-t)$ is a solution for Eq. (4) and then using (42), the motion equation system can be written as a function of the independent variable $-t$:

$$\begin{aligned} \frac{dx_1(-t)}{d(-t)} &= x_4(-t) \\ \frac{dx_2(-t)}{d(-t)} &= x_5(-t) \\ \frac{dx_3(-t)}{d(-t)} &= x_6(-t) \\ \frac{dx_4(-t)}{d(-t)} &= -2x_5(-t) + \frac{\partial U}{\partial x_1(-t)} \\ \frac{dx_5(-t)}{d(-t)} &= 2x_4(-t) + \frac{\partial U}{\partial x_2(-t)} \\ \frac{dx_6(-t)}{d(-t)} &= \frac{\partial U}{\partial x_3(-t)} \end{aligned} \tag{43}$$

The variation system for Eq. (43) will be

$$\frac{dZ(-t)}{d(-t)} = \begin{bmatrix} 0 & E \\ S(t) & -2K \end{bmatrix} Z(-t) \tag{44}$$

From (41) and (44), Eq. (40) becomes

$$\begin{aligned} HA(-t) + A(-t)H &= \begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix}\begin{bmatrix} 0 & E \\ S(t) & -2K \end{bmatrix} \\ &+ \begin{bmatrix} 0 & E \\ S(t) & 2K \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{45}$$

With condition (40) being fulfilled from Theorem 1, it follows that the monodromy matrix of the variation system (25) is reciprocal. The calculation of the monodromy matrix of system (25) implies evaluation of the terms of the matrix $S(t)$.

V. Symmetric Matrix $S(t)$

Because of its symmetry, matrix $S(t)$ can be written as

$$S(t) = \begin{bmatrix} U_{xx} & U_{xy} & U_{xz} \\ U_{xy} & U_{yy} & U_{yz} \\ U_{xz} & U_{yz} & U_{zz} \end{bmatrix} \tag{46}$$

An evaluation of the matrix terms can be provided by first-degree Taylor series expansions using the initial conditions (15). Keeping only the first-order terms the following expressions are obtained:

$$U_{xy}^1 = 3\left\{\frac{(1-\mu)(x_0+\mu)}{\left[(x_0+\mu)^2+z_0^2\right]^{\frac{5}{2}}} + \frac{\mu(x_0-1+\mu)}{\left[(x_0-1+\mu)^2+z_0^2\right]^{\frac{5}{2}}}\right\}\dot{y}_0$$

$$U_{yz}^1 = 3z_0\left\{\frac{1-\mu}{\left[(x_0+\mu)^2+z_0^2\right]^{\frac{5}{2}}} + \frac{\mu}{\left[(x_0-1+\mu)^2+z_0^2\right]^{\frac{5}{2}}}\right\}\dot{y}_0$$

$$\begin{aligned} U_{xx} &= U_{xx}^0, & U_{yy} &= U_{yy}^0, & U_{zz} &= U_{zz}^0 \\ U_{xy} &= tU_{xy}^1, & U_{xz} &= U_{xz}^0, & U_{yz} &= tU_{yz}^1 \end{aligned} \tag{47}$$

where index 0 represents the values at the initial time. It follows that

$$S(t) = \begin{bmatrix} U_{xx}^0 & 0 & U_{xz}^0 \\ 0 & U_{yy}^0 & 0 \\ U_{xz}^0 & 0 & U_{zz}^0 \end{bmatrix} + t\begin{bmatrix} 0 & U_{xy}^1 & 0 \\ U_{xy}^1 & 0 & U_{yz}^1 \\ 0 & U_{yz}^1 & 0 \end{bmatrix} \tag{48}$$

or

$$S(t) = S_0 + tS_1 \tag{49}$$

where S_0 and S_1 are defined by relation (48).

Taking into account (49), the coefficient matrix of the differential system can be written as

$$A(t) = \begin{bmatrix} 0 & E_3 \\ S_0 & 2K \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ tS_1 & 0 \end{bmatrix} \tag{50}$$

Let

$$B = \begin{bmatrix} 0 & E_3 \\ S_0 & 2K \end{bmatrix} \tag{51}$$

It follows that the monodromy matrix of B is $e^{B\omega}$. The eigenvalues of the matrix $e^{B\omega}$ have the form $e^{\lambda k^{\omega}}$, where λ_k are eigenvalues of the matrix B. The eigenvalues are the roots of the equation

$$\det|B - \lambda E_3| = \det\begin{vmatrix} -\lambda E_3 & E_3 \\ S_0 & 2K - \lambda E_3 \end{vmatrix} = 0 \tag{52}$$

which can be also written as

$$\det\left[\lambda^2 E_3 - \lambda(2K) - S_0\right] = 0 \tag{53}$$

By substituting $\lambda^2 = \rho$, Eq. (53) becomes

$$\begin{aligned} f(\rho) &= \rho^3 - \left(U_{xx}^0 + U_{yy}^0 + U_{zz}^0 - 4\right)\rho^2 \\ &+ \left[U_{xx}^0 U_{yy}^0 + U_{xx}^0 U_{zz}^0 + U_{yy}^0 U_{zz}^0 - \left(U_{xz}^0\right)^2 - 4U_{zz}^0\right]\rho \\ &- U_{xx}^0 U_{yy}^0 U_{zz}^0 + \left(U_{xz}^0\right)^2 U_{yy}^0 = 0 \end{aligned} \tag{54}$$

If the roots ρ_i ($i = 1, 2, 3$) are different, real, and negative, then the eigenvalues of the matrix B have the form $\pm ip_k$ ($k = 1, 2, 3$).

In this case the eigenvalues of the monodromy matrix B given by $e^{\pm ip_k}$ are placed on the unit circle. The expressions of the matrices $S(t)$ and S_1 show that, for sufficiently low values of the terms U_{xy}^1, U_{yz}^1, the symmetry matrices $S(t)$ and S_0 are quite similar, so that the same conclusion regarding matrices $A(t)$ and B defined by relations (50) and (51) holds.

VI. Analysis of Motion Stability on Halo Periodic Orbits

The results obtained and the formulated assumptions enable the solution of the stability analysis for the differential system using the following theorem:

Theorem 2. Assume the following conditions are fulfilled:

1) The monodromy matrix $W(\omega)$ of the system

$$\frac{dx}{dt} = Bx \tag{55}$$

has distinct eigenvalues placed on the unit circle.

2) The periodic system monodromy matrix $V(\omega)$ has a reciprocal characteristic equation

$$\frac{dx}{dt} = A(t)x \tag{56}$$

3) The expression

$$J = \int_0^{\omega} |A(t) - B^{\omega}| dt \tag{57}$$

is sufficiently small.

Then the solution of the differential system (56) is stable.

Demonstration. Because $W(t)$ and $V(t)$ are fundamental solutions of systems (56) and (55), respectively,

$$\frac{dV}{dt} = A(t)V, \qquad V(0) = E \tag{58a}$$

$$\frac{dW}{dt} = BW, \qquad W(0) = E \tag{58b}$$

Writing

$$\frac{dV}{dt} = BV(t) + [A(t) - B]V(t) \tag{59}$$

results in the solution

$$V(t) = W(t) + \int_0^t W(t)W^{-1}(s)[A(s) - B(s)]V(s)\,ds \tag{60}$$

As we have considered in the hypothesis, the monodromy matrix $W(\omega)$ has distinct eigenvalues placed on the unit circle. Thus its normal form is diagonal, with elements on the main diagonal having a modulus of 1.

It follows[14] that the sequences $W^n(\omega)$ and $W^{-n}(\omega)$ are bounded. The next relation has already been demonstrated:

$$W(t+\omega) = W(t)W(\omega) \tag{61}$$

Assuming that there is $m \in N$ so that $(m-1)\omega < t < m\omega$, from $t = t' + (m-1)\omega$, one can write

$$0 < t' < \omega \tag{62}$$

so that we shall deduce the bounding of the solution $W(t)$ considering its continuity.

It is also known that

$$W(t) = W[t' + (m-1)\omega] = W(t')W[(m-1)\omega] \tag{63}$$

Because

$$W[(m-1)\omega] = W(\omega + \cdots + \omega) = W^{m-1}(\omega) \tag{64}$$

it will finally follow that

$$W(t) = W(t')W^{m-1}(\omega) \tag{65}$$

Expression (65) shows the bounding of $W(t)$ with the condition that $W^{m-1}(\omega)$ is also bounded. For the solution $W^{-1}(t)$ of the adjoint (58b), it follows that

$$W^{-1}(t) = W^{-1}(t')W^{-(m-1)}(\omega) \tag{66}$$

By similar reasoning, from Eq. (66), there follows the bounding of the solution $W^{-1}(t)$.

An evaluation of the difference between solutions of systems (58a) and (58b) can be obtained by writing relation (60) in the form

$$V(t) - W(t) = \int_0^t W(t)W^{-1}(s)[A(s) - B(s)][V(s) - W(s)]\,\mathrm{d}s + \int_0^t W(t)W^{-1}(s)[A(s) - B(s)]W(s)\,\mathrm{d}s \tag{67}$$

Observing that $W(t)$ and $W^{-1}(t)$ are bounded for $t \in [(m-1)\omega, m\omega]$, particularly for $m = 1$, from (67), yields

$$\|V(\omega) - W(\omega)\| \le M_1 \int_0^\omega \|A(s) - B(s)\|\,\mathrm{d}s + M_2 \int_0^\omega \|A(s) - B(s)\|\|V(s) - W(s)\|\,\mathrm{d}s \tag{68}$$

From the inequality (68) it follows that

$$\|V(\omega) - W(\omega)\| \le M_1 \int_0^\omega \|A(s) - B(s)\|\,\mathrm{d}s \times \exp\left[M_z \int_0^\omega \|A(s) - B(s)\|\,\mathrm{d}s\right] \tag{69}$$

Taking into account hypothesis (3) from Eq. (69) it follows that the difference between $V(\omega)$ and $W(\omega)$ is sufficiently small. Consider the eigenvalues of the matrix $W(\omega)$ placed on the unit circle $|z| = 1$.

The distinct eigenvalues (s_k) will be surrounded with sufficiently small circles so that each one should include only a single eigenvalue and the circles should not intersect.

Because $W(\omega)$ and $V(\omega)$ are almost the same, each circle $(C_1), \ldots, (C_k)$ includes a single eigenvalue $\rho_1, \rho_2, \ldots, \rho_k$. If the solution $V(\omega)$ has a reciprocal characteristic equation, then $1/\rho_k$ are eigenvalues and together with these values admit also the roots $\rho_k = 1/\bar{\rho}_k$, symmetric with respect to the circle $|z| = 1$ and placed in (C_k).

Due to the fact that each circle was chosen to contain only a single eigenvalue, it follows that

$$\rho_k = 1/\bar{\rho}_k \tag{70}$$

$$|\rho_k| = 1 \tag{71}$$

Thus the eigenvalues of the matrix $V(\omega)$ are placed on the circle $|z| = 1$.

The hypothesis stated that the eigenvalues of $W(\omega)$ are distinct so the eigenvalues of $V(\omega)$ are also distinct.

Based on a known theorem,[13] the $W(t)$ solution of system (55) is stable (the eigenvalues have a modulus of 1 corresponding to one-dimension Jordan meshes), which demonstrates the stability of the solution $V(t)$ for system (56), observing that $W(\omega)$ and $V(\omega)$ have the same properties.

It is straightforward to prove that those systems for which the monodromy matrix characteristic equation is reciprocal exhibit only bounded stability, but not asymptotic stability, meaning that the multipliers (roots) are on the unit circle. Proving the stability of the system presumes the existence of the pure imaginary roots of the characteristic equation of B.

Using this result, an evaluation of the quantity $\int_0^\omega |A(s) - B|\,\mathrm{d}s$ can be obtained by formulating the following theorem:

Theorem 3. Let the matrix B given by (51) be constant, with eigenvalues having a null real part and $A(t)$ be the periodic matrix of the analyzed system. If $\int_0^\omega |A(s) - B|\,\mathrm{d}s$ is sufficiently small, the trivial solution of the differential system $\mathrm{d}x/\mathrm{d}t = A(t)x$ is stable.

Demonstration. Consider $x(t; x_0)$ the solution of the differential system, where x_0 represents the initial stage given at $t = 0$. The equations can be written in the form

$$\frac{\mathrm{d}[x(t; x_0)]}{\mathrm{d}t} = Bx(t; x_0) + [A(t) - B]x(t; x_0) \tag{72}$$

the solution being given by

$$x(t; x_0) = e^{Bt}x_0 + \int_0^t e^{B(t-s)}[A(s) - B]x(s; x_0)\,\mathrm{d}s \tag{73}$$

Using the known relation

$$x(t; x_0) = V(t)x_0 \tag{74}$$

formula (73) becomes

$$V(t) = e^{Bt} + \int_0^t e^{B(t-s)}[A(s) - B]V(s)\,\mathrm{d}s \tag{75}$$

According to the hypothesis the characteristic equation of the matrix B has null real part eigenvalues. Thus, we consider the solution structure of the constant coefficient linear system as

$$|e^{B(t-s)}| < M \tag{76}$$

Using (76) from relation (75) results in

$$|V(t)| \le M + M \int_0^t |A(s) - B||V(s)|\,\mathrm{d}s \tag{77}$$

From relation (77) it follows that

$$V(t) \le M \exp\left[M \int_0^t |A(s) - B|\,\mathrm{d}s\right] \tag{78}$$

Taking into account a result previously obtained, expressed by relation (65), it can be concluded that matrix $V(t)$ is bounded if the sequence $[V(\omega)]^k$, $k > 0$, is also bounded.

Thus, considering (78), one can write

$$|V(\omega)| \le M \exp\left[M \int_0^\omega |A(s) - B|\,\mathrm{d}s\right] < L \tag{79}$$

If

$$\int_0^\omega |A(s) - B|\,\mathrm{d}s < \frac{1}{M} \ln \frac{L}{M} \tag{80}$$

it follows that

$$|V(\omega)| < L \tag{81}$$

Expression (80) offers an evaluation of the quantity $\int_0^\omega |A(s) - B|\mathrm{d}s$ so that the analytical solution of the system can be established.

To conclude, it can be observed that the stability condition of the differential system is equivalent to distinct negative solutions of Eq. (54), $f(\rho) = 0$.

The result is then

$$f'(\rho) = -3\rho^2 + 2\rho\left(U^0_{xx} + U^0_{yy} + U^0_{zz} - 4\right) - \left[U^0_{xx}U^0_{yy} + U^0_{xx}U^0_{zz} + U^0_{yy}U^0_{zz} - \left(U^0_{xz}\right)^2 - 4U^0_{zz}\right] = 0 \tag{82}$$

Calculating, this yields

$$U^0_{xx} + U^0_{yy} + U^0_{zz} - 4 = 2 \tag{83}$$

Let ρ'_1 and ρ'_2 be the roots of Eq. (71), where we assumed $\rho'_1 < \rho'_2$. It also follows that for $\rho \in (\rho'_2, +\infty)$ there is the inequality $f'(\rho) < 0$. Thus for this interval the function $f(\rho)$ is decreasing so that, if $\rho'_2 < 0$, then

$$U^0_{xx}U^0_{yy} + U^0_{xx}U^0_{zz} + U^0_{yy}U^0_{zz} - \left(U^0_{xz}\right)^2 - 4U^0_{zz} > 0 \tag{84}$$

From Eqs. (83) and (84) it follows that $\rho'_1, \rho'_2 < 0$. The very same condition will also result if imposing another condition, namely, that the sum and the product of the roots of Eq. (82) should be negative.

Due to the fact that

$$\lim_{\rho \to -\infty} f(\rho) = +\infty, \qquad \lim_{\rho \to -\infty} f(\rho) = -\infty \tag{85}$$

we can conclude that the existence of real, distinct, negative roots

Table 1 Orbit 1: initial condition—corrections

Iteration no.			
1 integration	$x_0 = 0.805$	$z_0 = 0.3$	$\dot{y}_0 = 0.305$
	$y_1 = 0.10137 \times 10^{-11}$	$\dot{x}_1 = -0.31908 \times 10^{-2}$	$\dot{z}_1 = 0.32117 \times 10^{-2}$
	$\delta z_0 = 0.22487 \times 10^{-2}$	$\delta \dot{y}_0 = -0.15717 \times 10^{-2}$	$\delta(T^*/2) = 0.3129 \times 10^{-2}$
2 integration	$x_0 = 0.805$	$z_0 = 0.30224871$	$\dot{y}_0 = 0.30342827$
	$y_1 = 0.15108 \times 10^{-11}$	$\dot{x}_1 = 0.171508 \times 10^{-4}$	$\dot{z}_1 = 0.140741 \times 10^{-4}$
	$\delta z_0 = 0.70678 \times 10^{-5}$	$\delta \dot{y}_0 = -0.185369 \times 10^{-5}$	$\delta(T^*/2) = 0.120755 \times 10^{-4}$
3 integration	$x_0 = 0.805$	$z_0 = 0.30225578$	$\dot{y}_0 = 0.30342642$
	$y_1 = 0.187936 \times 10^{-11}$	$\dot{x}_1 = -0.5793704 \times 10^{-9}$	$\dot{z}_1 = 0.2590867 \times 10^{-9}$

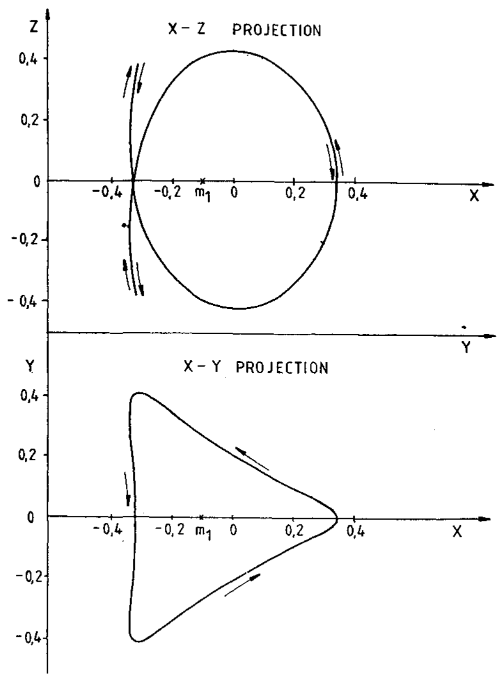

Fig. 1 Goudas case; $\mu = 0.1$.

of the equation $f(\rho) = 0$ is equivalent to simultaneously satisfying the relations

$$
\begin{aligned}
&0 < U^0_{xx}U^0_{yy} + U^0_{xx}U^0_{zz} + U^0_{yy}U^0_{zz} - \left(U^0_{xz}\right)^2 - 4U^0_{zz} < \tfrac{4}{3} \\
&U^0_{yy}\left[U^0_{xx}U^0_{zz} - \left(U^0_{xz}\right)^2\right] < 0 \\
&f(\rho_1') > 0 \\
&f(\rho_2') > 0
\end{aligned}
\tag{86}
$$

Thus $\rho_1 < \rho_1'$, $\rho_2 \in (\rho_1', \rho_2')$, $\rho_3 \in (\rho_2', 0)$ represent the stability conditions for the system of differential equations that was considered before.

The initial conditions x_i^0, corresponding to the periodic orbit family, will be determined by iteration, integrating system (23) for each fixed $\mu \in (0, 1)$.

The stability of these orbits necessarily assumes conditions (86) to be met.

VII. Numerical Application

In the following we analyze the periodic orbit family generated by the following initial conditions for $\mu = 0.1$:

$$x(0) = 0.34508360, \qquad y(0) = 0, \qquad z(0) = 0$$
$$\dot{x}(0) = 0, \qquad \dot{y}(0) = 0.25664780, \qquad \dot{z}(0) = 1.2911014$$

This case was considered by Goudas in 1963,[7] who obtained a three-dimensional periodic orbit in the form of a nondimensional ellipse.

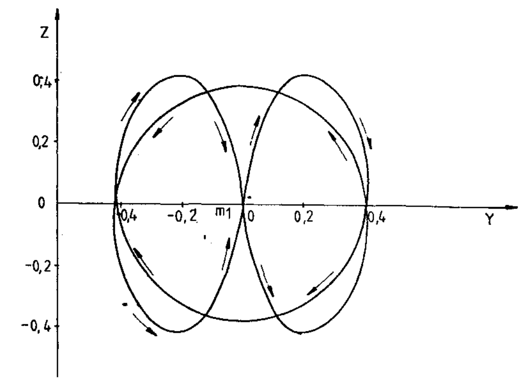

Fig. 2 Goudas case; $\mu = 0.1$; Y–Z projection: $X_0 = 0.3450836$, $Y_0 = 0$, $Z_0 = 0$; $T^*/2 = 3.11811726$: $\dot{X}_0 = 0$, $\dot{Y}_0 = 0.2566478$, $\dot{Z}_0 = 1.2911014$.

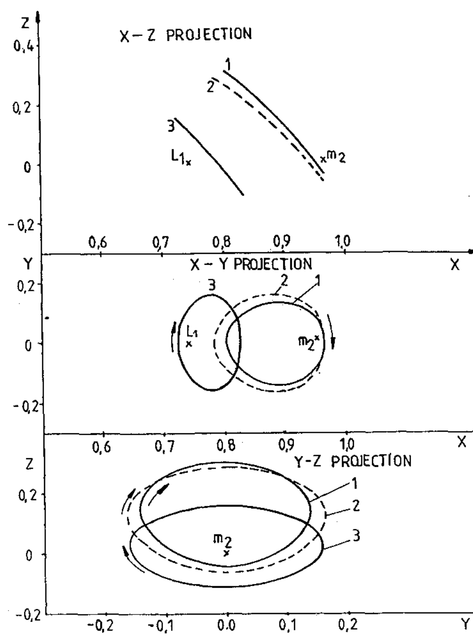

Fig. 3 L_1 family; $\mu = 0.04$; 1, $T^*/2 = 1.0064239$, $\dot{Y}_0 = 0.303426$; 2, $T^*/2 = 1.0733056$, $\dot{Y}_0 = 0.348231$; 3, $T^*/2 = 1.3480092$, $\dot{Y}_0 = 0.336578$; - - -, stable, ———, unstable.

For certain given values of $x(0)$ the corrections to the initial conditions have been determined so that the motion is periodic. Figures 1 and 2 represent Goudas case According with the stability criterion expressed by formula (86), the instability of this orbit results. Using the criterion defined by formula (86), the stability of the L_1 family orbits corresponding to $\mu = 0.04$ is determined. For this case of orbits we found the necessary corrections to the initial conditions so that the motion was periodic. The initial data generating the orbit that corresponds to the L_1 family suppose a fixed value of x_0. Let $x_1, y_1, z_1, \dot{x}_1, \dot{y}_1, \dot{z}_1$ be the final data. Four successive corrections on the initial data and on the half-period that are necessary to obtain the periodic orbit are shown in Tables 1–3.

Table 2 Orbit 2: initial condition—corrections

Iteration no.			
1 integration	$x_0 = 0.725$	$z_0 = 0.15200268$	$\dot{y}_0 = 0.33640719$
	$y_1 = 0.209358 \times 10^{-11}$	$\dot{x}_1 = 0.636306 \times 10^{-3}$	$\dot{z}_1 = 0.68925938 \times 10^{-4}$
	$\delta z_0 = 0.2039325 \times 10^{-3}$	$\delta \dot{y}_0 = 0.1631633 \times 10^{-3}$	$\delta(T^*/2) = 0.543366 \times 10^{-3}$
2 integration	$x_0 = 0.725$	$z_0 = 0.19220661$	$\dot{y}_0 = 0.33657035$
	$y_1 = 0.316810314 \times 10^{-11}$	$\dot{x}_1 = 0.97428308 \times 10^{-6}$	$\dot{z}_1 = 0.600205069 \times 10^{-8}$
	$\delta z_0 = 0.203932521 \times 10^{-3}$	$\delta \dot{y}_0 = 0.68726153 \times 10^{-6}$	$\delta(T^*/2) = 0.10062271 \times 10^{-5}$
3 integration	$x_0 = 0.725$	$z_0 = 0.15220719$	$\dot{y}_0 = 0.33657104$
	$y_1 = 0.26927119 \times 10^{-12}$	$\dot{x}_1 = 0.4392339 \times 10^{-11}$	$\dot{z}_1 = 0.713842744 \times 10^{-11}$

Table 3 Orbit 3: initial condition—corrections

Iteration no.			
1 integration	$x_0 = 0.78446$	$z_0 = 0.28672$	$\dot{y}_0 = 0.35$
	$y_1 = 0.80508 \times 10^{-11}$	$\dot{x}_1 = -0.5159734 \times 10^{-2}$	$\dot{z}_1 = 0.3067361 \times 10^{-2}$
	$\delta z_0 = 0.1841933 \times 10^{-2}$	$\delta \dot{y}_0 = -0.17763955 \times 10^{-2}$	$\delta(T^*/2) = 0.1956859 \times 10^{-2}$
2 integration	$x_0 = 0.78446$	$z_0 = 0.28856193$	$\dot{y}_0 = 0.3482236$
	$y_1 = 0.6330497 \times 10^{-12}$	$\dot{x}_1 = 0.20902117 \times 10^{-4}$	$\dot{z}_1 = -0.13110757 \times 10^{-4}$
	$\delta z_0 = -0.8010507 \times 10^{-5}$	$\delta \dot{y}_0 = 0.74494409 \times 10^{-5}$	$\delta(T^*/2) = -0.8646922 \times 10^{-5}$
3 integration	$x_0 = 0.78446$	$z_0 = 0.28855392$	$\dot{y}_0 = 0.34823105$
	$y_1 = 0.5747731 \times 10^{-11}$	$\dot{x}_1 = 0.362627856 \times 10^{-9}$	$\dot{z}_1 = -0.2323854 \times 10^{-9}$

The projection of the L_1 family orbit under consideration on the three planes is shown in Fig. 3.

VIII. Conclusions

Already existing papers[10,11] have approached the halo orbit stability problem using canonical transformations in which the matrix is symplectic, thus representing the necessary and sufficient condition. Taking into account that the characteristic equation of a symplectic matrix is reciprocal, the stability for a large class of orbits has been determined. In all previous papers the method of investigation for the stability of the periodic motion around L_1, L_2, L_3 points was the numerical application. Trying to avoid the great number of calculations and difficulties implied by the previously used methods, the present study offers an analytical solution of the stability of the halo periodic orbits. To determine the stability solution of a nonlinear differential system with periodic coefficients, we have suggested several stages: The system, whose stability is analyzed, belongs to a system class having the characteristic equation of the monodromy matrix reciprocal. In this class a fundamental property referring to the solution stability is demonstrated. For this condition to be met, we consider a constant coefficient system resulting from the linearization of the system analyzed. If the characteristic equation roots are distinct and placed on the unit circle, the orbit is stable. A numerical application was performed, fixing x_0 to the corresponding L_1, L_2, L_3 points with the other initial conditions determined so that the corresponding orbit was periodic. The stability of all orbits was analyzed.

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